Interaction of light with nanoresonators

- Photons and nanosystems
- Complex nanostructures
- Cold atoms, matter waves
- Biophotonic
- Optics & numerics (virtual reality)
Light interaction with nanoresonators

Near-field excitation

Far-field excitation
Resonators in nanophotonics

Individual resonator

- Si wire on SiO\(_2\)
- microcavity
- nanocavity
- picocavity

Field localization

Field enhancement

Hybridation of resonances

- Ag tip
- Photonic plasmonic hybrids
- Plasmonic molecules
- Colloidal films

- Biosensing
- Detectors
- Medical care
- Optical processing
- Metamaterials
- Quantum optics
- Subwavelength imaging
- Ultrafast-dynamics
- Random media
- Optical display
- .../...
Acoustic resonators: same physics

http://newt.phys.unsw.edu.au/jw/Helmholtz.html

\[ \tilde{F}(r) \exp(-i\omega t) \exp(-\Gamma t/2) \]

Modes are the essential concept governing the physics of resonant devices
Current standards in the analysis/design of nanoresonators

Numerical simulations repeated for each excitation

Physics described by just a few resonant modes

Modes are the essential concept governing the physics of resonant devices

King et al., ACS Nano 5, 7254 (2011)
Objective: modal formalism

Expected benefits:
• Straightforward interpretation of resonator response
• Superior numerical efficiency

\[
\begin{bmatrix}
H \\
E
\end{bmatrix} = \begin{bmatrix}
H_{\text{inc}} \\
E_{\text{inc}}
\end{bmatrix} + \sum_{m} \alpha_{m} \begin{bmatrix}
\tilde{H}_{m} \\
\tilde{E}_{m}
\end{bmatrix}
\]
Modal formalisms: waveguide versus cavity

Mature theory and advanced tools exist

“Light interaction with photonic and plasmonic resonances”

Truly-guided normal mode

Leaky resonance mode (quasinormal mode)
Two important issues for the expansion

- **Mode normalization**

Exponentially damped in time → exponentially diverging in space because of the minus sign

away from the resonator: \( r \to \infty : \exp[i\tilde{\omega}_m(t - r/c)] \)

- **Reconstruction problem: Completeness**

\[
\begin{bmatrix}
H \\
E
\end{bmatrix} = \begin{bmatrix}
H_{inc} \\
E_{inc}
\end{bmatrix} + \sum_m \alpha_m \begin{bmatrix}
\tilde{H}_m \\
\tilde{E}_m
\end{bmatrix}
\]

if we take all modes, do we get the exact solution? everywhere in space?
"Completeness and orthogonality of quasinormal modes in leaky optical cavities"


"Completeness and time-independent perturbation of the quasinormal modes of an absorptive and leaky cavity"


10 groups working in the world → so many progress that theory and numerical tools are "ready" for use

"Light interaction with photonic and plasmonic resonances"

P. Lalanne et al., Laser & Photonics Rev. 12, 1700113 (2018)
Drude metal, 600 nm period, TM, $\theta = 30^\circ$
\[
\begin{align*}
\mathbf{H} = & \mathbf{H}_{\text{inc}} + \sum_m \alpha_m \mathbf{H}_m \\
\end{align*}
\]
Introduction

1. The non-Hermitian nature of quasinormal modes
   - Reminder: Electromagnetism of Hermitian systems
   - Definition of quasinormal mode
   - Quasinormal mode computation
   - Normalization
   - Reconstruction

2. Numerical tests & examples
   - Continuous operator versus discretized operator
   - Numerical test: reconstruction and quenching
   - Fano analysis and temporal beating
   - QNM expansion of S-matrix

3. Physics of resonance
   - Cavity Q’s and V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
Hermitian electromagnetism

Photonic crystal
J.D. Joannopoulos et al.
(taken from chapter 2)

\[ \nabla \times \left( \frac{1}{\varepsilon(r)} \nabla \times H(r) \right) = \left( \frac{\omega}{c} \right)^2 H(r). \]  \hspace{1cm} (7)

This is the **master equation**. Together with the divergence equation (5), it tells us everything we need to know about \( H(r) \). Our strategy will be as follows: for a given structure \( \varepsilon(r) \), solve the master equation to find the modes \( H(r) \) and the corresponding frequencies.

\[ \hat{\Theta} H(r) = \left( \frac{\omega}{c} \right)^2 H(r). \]  \hspace{1cm} (10)

\[ \hat{\Theta} H(r) \triangleq \nabla \times \left( \frac{1}{\varepsilon(r)} \nabla \times H(r) \right). \]  \hspace{1cm} (11)

The eigenvectors \( H(r) \) are the spatial patterns of the harmonic modes, and the eigenvalues \( (\omega/c)^2 \) are proportional to the squared frequencies of those modes.
a special type of linear operator known as a **Hermitian operator**. In the coming sections we will develop these properties one by one. We conclude this section by showing what it means for an operator to be Hermitian. First, in analogy with the inner product of two wave functions, we define the inner product of two vector fields $\mathbf{F}(\mathbf{r})$ and $\mathbf{G}(\mathbf{r})$ as

\[
(\mathbf{F}, \mathbf{G}) \triangleq \int d^3\mathbf{r} \mathbf{F}^*(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r}),
\]

where "*" denotes complex conjugation. Note that a simple consequence of this definition is that $(\mathbf{F}, \mathbf{G}) = (\mathbf{G}, \mathbf{F})^*$ for any $\mathbf{F}$ and $\mathbf{G}$. Also note that $(\mathbf{F}, \mathbf{F})$ is always real and nonnegative, even if $\mathbf{F}$ itself is complex. In fact, if $\mathbf{F}(\mathbf{r})$ is a harmonic mode of our electromagnetic system, we can always set $(\mathbf{F}, \mathbf{F}) = 1$ by using our freedom to scale any mode by an overall multiplier. Given $\mathbf{F}'(\mathbf{r})$ with $(\mathbf{F}', \mathbf{F}') \neq 1$, create

\[
\mathbf{F}(\mathbf{r}) = \frac{\mathbf{F}'(\mathbf{r})}{\sqrt{(\mathbf{F}', \mathbf{F}')}}.
\]

From our previous discussion, $\mathbf{F}(\mathbf{r})$ is really the same mode as $\mathbf{F}'(\mathbf{r})$, since it differs only by an overall multiplier, but now (as the reader can easily verify) we have $(\mathbf{F}, \mathbf{F}) = 1$. We say that $\mathbf{F}(\mathbf{r})$ has been **normalized**. Normalized modes are very useful in formal arguments. If, however, one is interested in the physical energy of the field and not just its spatial profile, the overall multiplier is important.
that $\hat{\Theta}$ is Hermitian,\textsuperscript{10} we perform an integration by parts\textsuperscript{11} twice:

$$
(F, \hat{\Theta} G) = \int d^3 r F^* \cdot \nabla \times \left( \frac{1}{\varepsilon} \nabla \times G \right)
$$

$$
= \int d^3 r \left( \nabla \times F \right)^* \cdot \frac{1}{\varepsilon} \nabla \times G
$$

$$
= \int d^3 r \left[ \nabla \times \left( \frac{1}{\varepsilon} \nabla \times F \right) \right]^* \cdot G = (\hat{\Theta} F, G).
$$

(14)

In performing the integrations by parts, we neglected the surface terms that involve the values of the fields at the boundaries of integration. This is because in all cases of interest, one of two things will be true: either the fields decay to zero at large distances, or the fields are periodic in the region of integration. In either case, the surface terms vanish.
The operator is Hermitian, the eigenvalue are real

\[(H, \hat{\phi}H)^{\dagger} = (\omega^2 / c^2)^{\dagger} (H, H) = (\hat{\phi}H, H) = (\omega^2 / c^2) (H, H)\]
\[\implies (\omega^2 / c^2)^{\dagger} = (\omega^2 / c^2) \quad (16)\]

\[\omega_1^2 (H_2, H_1) = c^2 (H_2, \hat{\phi}H_1) = c^2 (\hat{\phi}H_2, H_1) = \omega_2^2 (H_2, H_1)\]
\[\implies (\omega_1^2 - \omega_2^2) (H_2, H_1) = 0 \quad (18)\]

If \(\omega_1 \neq \omega_2\), then we must have \((H_1, H_2) = 0\) and we say \(H_1\) and \(H_2\) are orthogonal modes. If two harmonic modes have equal frequencies \(\omega_1 = \omega_2\), then we say they are degenerate and they are not necessarily orthogonal. For two modes to be equation (26). By applying the perturbation procedure to equation (26), we obtain a simple formula for the frequency shift \(\Delta \omega\) that results from a small perturbation \(\Delta \varepsilon\) of the dielectric function:

\[\Delta \omega = -\frac{\omega}{2} \frac{\int d^3 r \Delta \varepsilon (r)|E(r)|^2}{\int d^3 r \varepsilon(r)|E(r)|^2} + O(\Delta \varepsilon^2) \quad (28)\]

In this equation, \(\omega\) and \(E\) are the frequency and the mode profile for the perfectly linear and lossless (unperturbed) dielectric function \(\varepsilon\). The error in this
Introduction

1. The non-Hermitian nature of quasinormal modes
   - Reminder: Electromagnetism of Hermitian systems
   - **Definition of quasinormal mode**
   - Quasinormal mode computation
   - Normalisation
   - Reconstruction

2. Numerical tests & examples

3. Physics of resonance
The quasinormal modes $[\hat{\mathbf{E}}(\mathbf{r}), \hat{\mathbf{H}}(\mathbf{r})] \exp i\tilde{\omega}t$ are factorized solutions of Maxwell’s equations without sources for a complex frequency $\tilde{\omega} = \omega_0 (1 - i/2Q)$

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
\nu^{-1}(\mathbf{r}, \tilde{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{E}} \\
\hat{\mathbf{H}}
\end{bmatrix} = \tilde{\omega}
\begin{bmatrix}
\hat{\mathbf{E}} \\
\hat{\mathbf{H}}
\end{bmatrix}
\]

+ outgoing wave conditions
Assume \([\tilde{E}(r), \tilde{H}(r), \tilde{\omega}]\) is a QNM

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
 i\varepsilon^{-1}(r, \tilde{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix} = \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
\]

+ outgoing wave conditions

Simply conjugate

\[
\begin{bmatrix}
0 & i\mu_0^{-1}\nabla \times \\
 -i\varepsilon^{-1}(r, \tilde{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E}^* \\
\tilde{H}^*
\end{bmatrix} = \tilde{\omega}^*
\begin{bmatrix}
\tilde{E}^* \\
\tilde{H}^*
\end{bmatrix}
\]

Hermitian symmetry of real Fourier transforms

\[\varepsilon^*(r, \omega) = \varepsilon(r, -\omega^*)\]

Multiply by \(-1\) and replace \(\varepsilon\)

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
 i\varepsilon^{-1}(r, -\tilde{\omega}^*)\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E}^* \\
\tilde{H}^*
\end{bmatrix} = -\tilde{\omega}^*
\begin{bmatrix}
\tilde{E}^* \\
\tilde{H}^*
\end{bmatrix}
\]

\[\frac{1}{2}\text{Re} \int (\tilde{E}^* \times \tilde{H}^{**}) \cdot dS = \frac{1}{2}\text{Re} \int (\tilde{E} \times \tilde{H}^*) \cdot dS\]

Conclude

\([\tilde{E}^*(r), \tilde{H}^*(r), -\tilde{\omega}^*]\) is also a QNM
Symmetry of the eigenspectrum

\[ \exp(-i\omega t) \] notations

[\[\tilde{E}^*(r), \tilde{H}^*(r), -\omega^*\]]

[\[\tilde{E}(r), \tilde{H}(r), \omega\]]
Introduction

1. The non-Hermitian nature of quasinormal modes
   - Reminder: Electromagnetism of Hermitian systems
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   - Quasinormal mode computation
   - Normalisation
   - Reconstruction

2. Numerical tests & examples

3. Physics of resonance
how to compute QNMs?

$$
\begin{bmatrix}
0 & -i \mu_0^{-1} \nabla \times \\
\epsilon^{-1} (r, \tilde{\omega}) \nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix} = \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
$$

+ outgoing wave conditions
QNM solvers (non dispersive case)

Assume first the simple case of non-dispersive materials

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
\varepsilon^{-1}(\mathbf{r}, \mathbf{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{E}} \\
\hat{\mathbf{H}}
\end{bmatrix}
= \omega
\begin{bmatrix}
\hat{\mathbf{E}} \\
\hat{\mathbf{H}}
\end{bmatrix}
\]

+ outgoing wave conditions

- You face a standard linear eigenproblem (like for photonic crystals)
- You discretize the operator
- You compute the eigenvectors
- Even commercial software does it for you
Assume general case of dispersive materials

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
i\varepsilon^{-1}(\mathbf{r}, \tilde{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{\mathbf{E}} \\
\tilde{\mathbf{H}}
\end{bmatrix}
= \tilde{\omega}
\begin{bmatrix}
\tilde{\mathbf{E}} \\
\tilde{\mathbf{H}}
\end{bmatrix}
\]

+ outgoing wave conditions

First problem: how are you defining the permittivity at a complex frequency?
N-pole Drude Lorentz permittivity

Fitting Parameters for Silver – Palik data [1]: We fit the Palik data for silver using one plasmonic Drude model with damping plus six modified Lorentz terms,

\[ \varepsilon_{Ag}(\omega) = 1 - \omega_0^2 / (\omega^2 + i\omega\gamma_0) + \sum_{i=1}^{6} \omega_{p,i}^2 / (\omega^2 - \omega_{p,i}^2 + i\omega\gamma_i) \]

Assume general case of dispersive material

\[
\begin{bmatrix}
0 & \frac{-i\mu_0^{-1}\nabla}{\nabla} \\
\frac{i\varepsilon^{-1}(\mathbf{r}, \tilde{\omega})\nabla}{\nabla} & 0
\end{bmatrix}\begin{bmatrix}
\hat{E} \\
\hat{H}
\end{bmatrix} = \tilde{\omega}\begin{bmatrix}
\hat{E} \\
\hat{H}
\end{bmatrix}
\]

+ outgoing wave conditions

- First problem: how are you defining the permittivity at a complex frequency

- Second problem appears: to solve a nonlinear eigenproblem
Method (1): pole computation

- **GENERAL** (can be used with any Maxwell’s solver method), **VERY** simple

- **Drawbacks**: iterative computation (an initial knowledge of the mode frequency is required $\Rightarrow$ Excellent to compute a few dominant modes)

"Efficient and intuitive method for the analysis of light scattering by a resonant nanostructure"
Method (2): linearization with auxiliary fields

- QNM eigenvalue equation

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \\
\varepsilon^{-1}(r, \tilde{\omega})\nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix} = \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
\]

- Linearized QNM eigenvalue equation: auxiliary field method

\[N\text{-pole Lorentz-Drude permittivity: } \varepsilon(\omega)/\varepsilon_\infty = 1 - \sum_{i=1}^{N} \omega_p^2/(\omega^2 - \omega_p^2 + i\omega \gamma_i)\]

- Advantages:
  - Access to all eigenvalues in a “single” step, no preconditioning
- Drawbacks:
  - Matrix size is larger

References:

Examples

"Rigorous modal analysis of plasmonic resonances"
Introduction

1. The non-Hermitian nature of quasinormal modes
   – Reminder: Electromagnetism of Hermitian systems
   – Definition of quasinormal mode
   – Quasinormal mode computation
   – Normalization
   – Reconstruction

2. Numerical tests & examples

3. Physics of resonance
QNM normalization: main steps

- **Initial works for high Q modes (perturbation of perfect Hermitian system)**

  \[ \iiint \left[ \epsilon |\vec{E}_m|^2 + \mu_0 |\vec{H}_m|^2 \right] d^3r = 1 \]

- **Correct normalization of divergent QNMs**
  1D system and Mie resonance

  General case: arbitrary resonators

  \[ \iiint \left[ \frac{\partial\omega\epsilon}{\partial \omega} \vec{E}_m \cdot \vec{E}_m - \mu_0 \vec{H}_m \cdot \vec{H}_m \right] d^3r = 1 \]

- **Helpful critical papers**
  Critical analysis of some popular methods

  Critical comparison of difference methods
Complex coordinate transform (PML)

\[ X = \alpha (1 + i m) x \]
\[ Y = \alpha (1 + i m) y \]
\[ Z = \alpha (1 + i m) z \]

Normalization issue: solved in 2013 for general case

\[ N_m = \iiint \left( \frac{\partial \omega \varepsilon}{\partial \omega} \hat{E}_m^2 - \mu_0 \hat{H}_m^2 \right) d^3r \]

- proof that \( N_m \) is invariant under any space coordinate transforms, including complex ones
- Evidence that \( N_m \) well defines the excitation coefficients \( \alpha_m \)'s
- Numerical method to calculate \( N_m \), by computing the integral in real space and in the PML.

Silver nanosphere (radius 100 nm) in air
\[ \lambda = 0.6007 + 0.2344i \]

PML defines an analytic continuation of \( N_m \) in the complex plane \( r' + i r'' \) for \( r > r_0 \)
\[ R_{\text{real space}} \]
\[
N_m = (0.1947 - 0.2735i) + (0.8053 + 0.2736i)
\]
\[ = 1.0000 \]

\[ R_{\text{PML}} \]
\[
N_m = 1.0000
\]

**Freeware for normalizing: QNMPole**

- **GENERAL** (can be used with any Mawell’s solver method), VERY simple: \( \lim_{\omega \to \tilde{\omega}_m} E(r, \omega) = \frac{-i J \cdot \tilde{E}_m(r)}{\omega - \tilde{\omega}_m} \tilde{E}_m(r) \). Freeware **QNMPole** available @ our group webpage

- **Drawbacks**: iterative computation (an initial approximate value of the mode frequency is required \( \Rightarrow \) Excellent to compute a few dominant modes)

"Efficient and intuitive method for the analysis of light scattering by a resonant nanostructure"
Freeware for normalizing: QNMEig

\[
\begin{bmatrix}
0 & -i\mu_0^{-1}\nabla \times \nabla \times \\
0 & 0 & 0 & 0 \nabla \times \\
0 & i\omega_p^2\varepsilon_\infty & 0 & -i\varepsilon_\infty^{-1} \nabla \times \\
0 & -i\omega_0^2 & -i\gamma & i
\end{bmatrix}
\begin{bmatrix}
\mathbf{E} \\
\mathbf{H} \\
\mathbf{P} \\
\mathbf{J}
\end{bmatrix} = \tilde{\omega}
\begin{bmatrix}
\mathbf{E} \\
\mathbf{H} \\
\mathbf{P} \\
\mathbf{J}
\end{bmatrix}
\]

polarization current

Freeware QNMEig available @ our group webpage

"Rigorous modal analysis of plasmonic resonances"
plasmonic nanocavities .................................................. p 2

\[
\varepsilon_{\text{Ag}} = \varepsilon_{\infty} - \frac{\omega_{p,\text{Ag}}^2}{\omega(\omega + i\gamma)}
\]

Au substrate

\[
\varepsilon_{\text{Au}} = \varepsilon_{\infty} - \frac{\omega_{p}^2}{\omega(\omega + i\gamma_{\text{Au}})} - \frac{A\omega^2}{\omega^2 - \omega_1^2 + i\omega\gamma_1}
\]

photonic microcavities ..................................................... p 6

(gratings and crystals .................................................. p 8

(no dispersion)

\[
\varepsilon_{\text{Ag}} = \varepsilon_{\infty} - \frac{\omega_{p,\text{Ag}}^2}{\omega(\omega + i\gamma)}
\]

slit array in an Ag membrane \(\tilde{\omega}(\theta)\)

plasmonic crystal \(\tilde{\omega}(k)\)
Introduction

1. The non-Hermitian nature of quasinormal modes
   - Reminder: Electromagnetism of Hermitian systems
   - Definition of quasinormal mode
   - Quasinormal mode computation
   - Normalization
   - Reconstruction

2. Numerical tests & examples

3. Physics of resonance
Reconstruction

$$\begin{bmatrix} \mathbf{H} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{inc} \\ \mathbf{E}_{inc} \end{bmatrix} + \sum_m \alpha_m \begin{bmatrix} \tilde{\mathbf{H}}_m \\ \tilde{\mathbf{E}}_m \end{bmatrix}$$

$$\varepsilon(\mathbf{r}, \omega) = \varepsilon_b(\mathbf{r}, \omega) + \Delta\varepsilon(\mathbf{r}, \omega)$$

- nanoparticle in air
- nanoparticle on a substrate
- More complicated system
Reconstruction: dispersive case

\[
\begin{bmatrix}
H \\
E
\end{bmatrix} = \begin{bmatrix}
H_{inc} \\
E_{inc}
\end{bmatrix} + \sum_{m} \alpha_m \begin{bmatrix}
\tilde{H}_m \\
\tilde{E}_m
\end{bmatrix}
\]

The \( \alpha_m \)'s are known analytically first-principles derivations (Maxwell's equation + unconjugate form of Lorentz reciprocity theorem)

\[
\frac{\varepsilon(\omega)}{\varepsilon_{\infty}} = 1 - \sum_{i=1}^{N} \frac{\omega_{p,i}^2}{(\omega^2 - \omega_{p,i}^2 + i\omega \gamma_i)}
\]

\[
\alpha_m(\omega) = \frac{\tilde{\omega}_m}{\tilde{\omega}_m - \omega} \int \int \int_{V_{res}} (\varepsilon(\mathbf{r}, \tilde{\omega}_m) - \varepsilon_b) \tilde{E}_m(\mathbf{r}) \cdot E_{inc}(\mathbf{r}, \omega) \, d^3 \mathbf{r}
\]

\[
+ \int \int \int_{V_{res}} (\varepsilon_b - \varepsilon_{\infty}) \tilde{E}_m(\mathbf{r}) \cdot E_{inc}(\mathbf{r}, \omega) \, d^3 \mathbf{r}
\]

each term is a weighted classical overlap between the incident field \( E_{inc} \) and the mode \( \tilde{E}_m \)

"Rigorous modal analysis of plasmonic resonances"
For the specific case of non dispersive materials, the previous equation leads to

\[ \alpha_m(\omega) = -\frac{\omega}{\omega - \tilde{\omega}_m} \iiint_{V_{res}} \Delta \epsilon(\mathbf{r}) E_{inc}(\mathbf{r}, \omega) \cdot \tilde{E}_m(\mathbf{r}) d^3 \mathbf{r} \]

classical overlap between the incident field \( E_{inc} \) and the mode \( \tilde{E}_m \).
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & applications
   - Continuous operator versus discretized operator
   - Numerical test: reconstruction and quenching
   - Fano analysis and temporal beating
   - QNM expansion of S-matrix

3. Physics of Q’s and V’s
Is the QNM basis complete?

P. Lalanne et al., Laser Photonics Rev. 12, 1700113 (2008) "Light interaction with photonic and plasmonic resonances"

\( \mathcal{F} \): Spectral interval for which the discretization is accurate and the outgoing-wave condition is faithfully matched
Continuous versus discretized operator

- Continuous operator & open space
- Discretized operator and closed space

Is the QNM basis complete?

P. Lalanne et al., Laser Photonics Rev. 12, 1700113 (2008) "Light interaction with photonic and plasmonic resonances"

The basis is complete
metal sphere:
\[ \varepsilon = 3 - \frac{\omega_p^2}{\omega^2 + i\omega \gamma} \]
\[ \gamma = 0.02 \omega_p \]
\[ \lambda_p = \frac{2\pi c}{\omega_p} = 138 \text{ nm} \]
60 nm diameter

Spectrum

\[ \ln(\tilde{\omega}_m) / \omega_p \]
\[ \text{Re}(\tilde{\omega}_m) / \omega_p \]

Reconstruction

Extinction cross section \[ \lambda_p^2 \]

Convergence

Mean absolute error of extinction cross section \[ \lambda_p^2 \]

Mode truncation rank, \( M \)
metal sphere:
\[ \varepsilon = 3 - \frac{\omega_p^2}{\omega^2 + i \omega \gamma} \]
\[ \gamma = 0.02 \omega_p \]
\[ \lambda_p = \frac{2 \pi c}{\omega_p} = 138 \text{ nm} \]
60 nm diameter

Spectrum

Reconstruction

Convergence
metal sphere:
\[ \varepsilon = 3 - \frac{\omega^2}{\omega^2 + i\omega\gamma} \]
\[ \gamma = 0.02\omega_p \]
\[ \lambda_p = \frac{2\pi c}{\omega_p} = 138 \text{ nm} \]
60 nm diameter

Spectrum

Reconstruction

Convergence

Movie showing the convergence & reconstruction is missing in the pdf version
Silver permittivity
\[ \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma} \]
\[ \gamma = 0.0023\omega_p \]
\[ \omega_p \sim 138 \text{ nm} \]

The solver implemented with COMSOL multiphysics is available @ our group webpage

"Rigorous modal analysis of plasmonic resonances"
metal bowtie:

\[ \varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + \omega \gamma} \]

\[ \gamma = 0.0023 \omega_p \]

\[ \lambda_p = \frac{2 \varepsilon \omega}{\omega_p} = 138 \text{ nm} \]

Reconstruction

Extinction cross section \([ \lambda_p^2 ]\)

Convergence

Mean absolute error of extinction cross section \([ \lambda_p^2 ]\)

Mode truncation rank, \(M\)
metal bowtie:
\[ \varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + \omega \gamma} \]
\[ \gamma = 0.0023 \omega_p \]
\[ \lambda_p = \frac{2\pi\varepsilon \omega_p}{\omega_e} = 138 \text{ nm} \]

Reconstruction

Extinction cross section \([\lambda_p^2]\)

\[ \frac{\omega}{\omega_p} \]

Convergence

Mean absolute error of extinction cross section \([\lambda_p^2]\)

Mode truncation rank, \(M\)
metal bowtie:
\[ \varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + \gamma^2} \]
\[ \gamma = 0.0023 \omega_p \]
\[ \lambda_p = \frac{2\pi c}{\omega_p} = 138 \text{ nm} \]

**Spectrum**

- + used modes
- (10 modes)

**Reconstruction**

**Convergence**

Mean absolute error of extinction cross section \[ [\chi_p^2] \]

Mode truncation rank, \( M \)
metal bowtie:
\[ \varepsilon = 1 - \frac{\omega^2}{\omega^2 + \omega \gamma} \]
\[ \gamma = 0.0023 \omega_p \]
\[ \lambda_p = \frac{2 \pi c}{\omega_p} = 138 \text{ nm} \]

Reconstruction

Extinction cross section \[ \langle \lambda_p^2 \rangle \]

\[ \frac{\omega}{\omega_p} \]

COMSOL data
modal method

Convergence

Mean absolute error of extinction cross section \[ \langle \lambda_p^2 \rangle \]

Mode truncation rank, \( M \)

Movie showing the convergence & reconstruction is missing in the pdf version
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples
   - Continuous operator versus discretized operator
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3. Physics of Q’s and V’s
Push the method to the limit: Quenching

QNM expansion of the total decay rate:

\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re} \left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentzian}(\omega - \tilde{\Omega}_m) \left[ 1 + 2Q_m \frac{\omega - \tilde{\Omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right]
\]

dipole moment: \( p\hat{u} \)

quenching occurs in a tiny localized volume just near the source

intrinsic field maps are independent of the source and delocalized
Push the method to the limit: Quenching

\[ \frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re}\left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentzian}(\omega - \tilde{\Omega}_m) \left[ 1 + 2Q_m \frac{\omega - \tilde{\Omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right] \]

"Rigorous modal analysis of plasmonic resonances"
Push the method to the limit: Quenching

Distribution of the absorbed power density

"Rigorous modal analysis of plasmonic resonances"
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & applications
   - Continuous operator versus discretized operator
   - Numerical test: reconstruction and quenching
   - Fano analysis and temporal beating
   - QNM expansion of S-matrix

3. Physics of Q’s and V’s
QNM analysis of Fano shapes

\[
\begin{bmatrix} H \\ E \end{bmatrix} = \begin{bmatrix} H_{inc} \\ E_{inc} \end{bmatrix} + \sum_{m} \alpha_m \begin{bmatrix} \tilde{H}_m \\ \tilde{E}_m \end{bmatrix}
\]

Just take two modes: \( \mathbf{E}_{sca} = \alpha_1 \tilde{\mathbf{E}}_1 + \alpha_2 \tilde{\mathbf{E}}_2 \)

\[
= \frac{\tilde{\omega}_1}{\tilde{\omega}_1 - \omega} \langle \tilde{\mathbf{E}}_1^* | \Delta \varepsilon | \mathbf{E}_{inc}(\omega) \rangle \tilde{\mathbf{E}}_1 + \frac{\tilde{\omega}_2}{\tilde{\omega}_2 - \omega} \langle \tilde{\mathbf{E}}_2^* | \Delta \varepsilon | \mathbf{E}_{inc}(\omega) \rangle \tilde{\mathbf{E}}_2
\]
Fano shapes: grating case

Specular reflectance vs wavelength (µm)
Fano shapes: grating case
Fano in the temporal domain

Fano response in frequency domain $\rightarrow$ Beating in time domain

\[
\begin{bmatrix}
H_{sca} \\
E_{sca}
\end{bmatrix} = \sum_m \beta_m(t) \begin{bmatrix}
\tilde{H}_m(r) \\
\tilde{E}_m(r)
\end{bmatrix}
\]

$\omega_0 = 2.9 \times 10^{15}$ rad/s
10 fs pulse duration

“Plasmon-Induced Transparency in Metamaterials”
3 fundamental modes previously computed

Accurate mode solver

\[
\left| \frac{\omega_{\text{iterative}} - \omega_{\text{solver}}}{\omega_{\text{iterative}} + \omega_{\text{solver}}} \right| \approx 10^{-14}
\]
Resonance mode expansion in time domain

\[ \beta_1(t) \tilde{E}_1(r) \]

\[ \beta_2(t) \tilde{E}_2(r) \]

\[ \beta_3(t) \tilde{E}_3(r) \]
Modal analysis of the ultrafast dynamics of nanoresonators

\[
\begin{bmatrix}
H_{sca} \\
E_{sca}
\end{bmatrix} = \sum_{m} \beta_{m}(t) \begin{bmatrix}
\tilde{H}_{m}(r) \\
\tilde{E}_{m}(r)
\end{bmatrix}
\]

Faggiani et al., ACS Photonics 4, 897 (2017).
"Modal analysis of the ultrafast dynamics of nanoresonators"
Amplitude difference

Dephasing

FDTD method
- 1 week computation
- Changes in the driving field properties (duration, central frequency, polarization, incidence angle) \(\Rightarrow\) totally new computation

Modal expansion
- Mode computation (few minutes per mode)
- Semi-analytical formalism
  \(\Rightarrow\) Changes in the driving field properties are computed in few seconds

Accuracy of the computational results

Faggiani et al., ACS Photonics 4, 897 (2017).
"Modal analysis of the ultrafast dynamics of nanoresonators"
Matlab demo
Frequency domain

Much faster than FDTD
~10 mins CPU times

Time domain

Field at gap center

"Rigorous modal analysis of plasmonic resonances"
Temporal domain analysis

"Rigorous modal analysis of plasmonic resonances"
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples
   - Continuous operator versus discretized operator
   - Numerical test: reconstruction and quenching
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   - QNM expansion of S-matrix

3. Physics of Q’s and V’s
Metal nanoparticle on a slab waveguide
\[ \sigma_{\text{sc}} / \lambda_p^2 \]

\[ /\omega_p \]

QNMs only

QNMs + PML-modes

Convergence performance

\[ \langle |\sigma_{\text{ext}} - \sigma_{\text{COM}}| \rangle_{\text{avg.}} \]

Truncation rank, \( M \)
Antenna analysis

\[ \omega, \mathbf{k}, \text{TE, TM} \quad \rightarrow \quad \omega, \mathbf{k}', \text{TE, TM} \]

Linearity of Maxwell’s equations → \[ A_S = S_{\omega, \mathbf{k}', \text{TE, TM}} A_i \]
Metal nanoparticle on a slab waveguide

Radiation diagram into guided modes

Free-space radiation diagram (S-matrix)

Metal nanocube

Free-space radiation diagram

SPP radiation diagram

SPP Excitation Pattern

Faggiani et al., ACS Photon. 2, 1739 (2015)

Ag

Au

g=8 nm
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
B10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. Purcell, Harvard University.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

\[ A_\nu = \frac{(8\pi^2/c^4)k_B(8\pi^2\mu^3/3\hbar^3)}{\sec^{-1}} \]

is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°C, for \( \nu = 10^7 \) sec.\(^{-1} \), \( \mu = 1 \) nuclear magneton, the corresponding relaxation time would be \( 5 \times 10^{11} \) seconds! However, for a system coupled to a resonant electrical circuit, the factor \( 8\pi^2/c^4 \) no longer gives correctly the number of radiation oscillators per unit volume, in unit frequency range, there being now one oscillator in the frequency range \( \nu/Q \) associated with the circuit. The spontaneous emission probability is reduced, and the relaxation time reduced,

\[ f = \frac{3Q\lambda^3}{4\pi^2V} = \frac{\tau_0}{\tau} \]

where \( V \) is the volume of the resonator, \( \lambda \) is a characteristic of the circuit so that \( V \sim a^3 \), and if \( \delta \) is the skin-depth at frequency \( \nu \), \( f \sim \lambda^3/a^3\delta \). For a non-resonant circuit \( f \sim \lambda^3/a^3 \), and for \( a < \delta \) it can be shown that \( f \sim \lambda^3/a^3\delta \).

If small metallic particles, of diameter \( 10^{-8} \) cm are mixed with a nuclear-magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for \( \nu = 10^7 \) sec.\(^{-1} \).
Classical Purcell effect

1946 Purcell paper

B10. Spontaneous Emission Probabilities at Radio Frequencies. E. M. Purcell, Harvard University.—For nuclear magnetic moment transitions at radio frequencies the probability of spontaneous emission, computed from

\[ A_r = \frac{8\pi^3}{c^3} h \nu \left( \frac{8\pi^4 \mu^2/3 \hbar^3}{\mu m} \right) \text{ sec}^{-1}, \]

is so small that this process is not effective in bringing a spin system into thermal equilibrium with its surroundings. At 300°K, for \( \nu = 10^7 \text{ sec}^{-1} \), \( \mu = 1 \) nuclear magneton, the corresponding relaxation time would be \( 5 \times 10^4 \) seconds! However, for a system coupled to a resonant electrical circuit, the factor \( 8\pi^3/c^3 \) no longer gives correctly the number of radiation oscillators per unit volume, in unit frequency range, there being now one oscillator in the frequency range \( \nu/Q \) associated with the circuit. The spontaneous emission probability is then reduced by \( V/Q \), where \( V \) is the volume of the resonator. \( \lambda \) is a characteristic of the circuit so that \( V \sim \lambda^2 \), and if \( \delta \) is the skin-depth at frequency \( \nu \), \( f \sim \lambda^2/\lambda^2 \). For a non-resonant circuit \( f \sim \lambda^2/\lambda^2 \), and for \( \delta < \lambda \) it can be shown that \( f \sim \lambda^2/\lambda^2 \).

If small metallic particles, of diameter \( 10^{-8} \) cm are mixed with a nuclear-magnetic medium at room temperature, spontaneous emission should establish thermal equilibrium in a time of the order of minutes, for \( \nu = 10^7 \text{ sec}^{-1} \).

f = \frac{3Q\lambda^3}{4\pi^2V} = \frac{P}{P_0}
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
"The amplitude are normalized such that \(|a|^2\) is the stored energy in the cavity"

\[
\text{Stored energy in the cavity} = \iiint_{\Omega} \left[ \varepsilon(\tilde{\omega}_m)|\tilde{E}|^2 + \mu(\tilde{\omega}_m)|\tilde{H}|^2 \right] dV
\]

What is \(V\)?
Can \(V\) be the entire space?

PhC microcavities

Plasmonic antenna

Can I define energy for a metal?
The Q’s

The non dispersive case

power dissipated by radiation

\[
\frac{1}{2} \iiint_{\Sigma} \text{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} + \frac{\text{Re} \, \tilde{\omega}}{2} \iiint_{V} \left[ \text{Im}[\varepsilon(\tilde{\omega})] |\mathbf{E}|^2 + \text{Im}[\mu(\tilde{\omega})] |\mathbf{H}|^2 \right] dV
\]

power dissipated by absorption

\[
\frac{\text{Im} \, \tilde{\omega}}{2} \iiint_{V} \left[ \text{Re}[\varepsilon(\tilde{\omega})] |\mathbf{E}|^2 + \text{Re}[\mu(\tilde{\omega})] |\mathbf{H}|^2 \right] dV
\]

time-averaged electromagnetic energy stored in the volume \( V \)

P. Lalanne et al., Laser & Photonics Rev. 12, 1700113 (2018)
"Light interaction with photonic and plasmonic resonances"
The Q’s

The non dispersive case

\[
Q = \frac{\text{Re } \tilde{\omega} \times \text{Stored energy in the box}}{\text{Power loss}}
\]

(by definition, Jackson’s book)

\[
Q = -\frac{\text{Re } \tilde{\omega}}{2 \text{ Im } \tilde{\omega}}
\]

(deduced from previous eq.)
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
The V’s

- Hermitian physics

\[ V(r_0) = \frac{\iiint \left[ \varepsilon |\vec{E}|^2 + \mu_0 |\vec{H}|^2 \right] d^3r}{2\varepsilon(r_0)|\vec{E}(r_0)|^2} \]

"could" be considered as valid for \( Q \gg 1 \)

\( V \) is real and measures the mode spatial extent (the cavity coupling strength)


- non-Hermitian physics

\[ \tilde{V} = \frac{\iiint (\varepsilon\vec{E}^2 - \mu_0\vec{H}^2) d^3r}{2\varepsilon(r_0)|\vec{E}(r_0)|^2} \]

\( \tilde{V} \) is complex, only its real part measures the mode spatial extent; what is the imaginary part?

QNMs in the limit of Hermitian physics \((Q \to \infty)\)

\[
\begin{bmatrix}
0 & -i \mu_0^{-1} \nabla \times \\
-i \varepsilon^{-1} (r, \tilde{\omega}) \nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
= \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -i \mu_0^{-1} \nabla \times \\
-\varepsilon^{-1} (r, \tilde{\omega}) \nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
= \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -\mu_0^{-1} \nabla \times \\
-\varepsilon^{-1} (r, \tilde{\omega}) \nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
= \tilde{\omega}
\begin{bmatrix}
\tilde{E} \\
\tilde{H}
\end{bmatrix}
\]

\[
\tilde{V}(r_0) = \frac{\iiint (\varepsilon \tilde{E}^2 - \mu_0 \tilde{H}^2) d^3r}{2\varepsilon(r_0) \tilde{E}^2(r_0)} \quad \iff \quad V(r_0) = \frac{\iiint [\varepsilon |\tilde{E}|^2 + \mu_0 |\tilde{H}|^2] d^3r}{2\varepsilon(r_0) |\tilde{E}(r_0)|^2}
\]
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
Cavity perturbation theory

Closed-form expression for $\Delta \tilde{\omega}$?
• Microwaves


• Photonic cavities

\[
\frac{\Delta \omega}{\omega} = -\frac{\alpha_e}{4} \frac{|\mathcal{E}|^2}{\langle W \rangle}
\]

\[
\langle W \rangle = \frac{1}{16\pi} \int_{V_c} (|\mathcal{E}(r)|^2 + |\mathcal{H}(r)|^2) dv
\]


F. Koenderink et al., PRL 95, 153904 (2005)
L. Ramunno and S. Hughes, PRB 79, 161303(R) (2009)

• plasmon sensors

\[
\Delta \omega_m = -\alpha_{NP} \frac{d\omega}{d\varepsilon_{ca}} \frac{|E_m(r_{NP})|^2}{\int_{cavity} \int \int |E_m(r)|^2 d^3r}
\]

$$\frac{\Delta \tilde{\omega}}{\tilde{\omega}} \approx \frac{-\alpha \varepsilon(r_0)|\tilde{E}(r_0)|^2}{\iint \left[ \varepsilon|\tilde{E}|^2 + \mu_0|\tilde{H}|^2 \right] d^3r} \equiv \frac{-\alpha}{2V(r_0)}$$

Hermitian formula

\[\frac{\Delta \tilde{\omega}}{\tilde{\omega}} \approx \frac{-\alpha \varepsilon(r_0)\tilde{E}^2}{\iint \left[ \varepsilon\tilde{E}^2 - \mu_0\tilde{H}^2 \right] d^3r} \equiv \frac{-\alpha}{2\tilde{V}(r_0)}\]

J. Yang et al., Nano Lett. 15, 3439 (2015)
Plasmonic cavity

Scaling factor = 1

Real \((\Delta \lambda)\) [nm]

\[ \approx \frac{\lambda}{15} \]

2Im \((\Delta \lambda)\) [nm]

\[ \approx \frac{\lambda}{4} \]

- Exact by solving the perturbed case

- Previous state-of-the-art
  "A Universal Law for Plasmon Resonance Shift in Biosensing"

  "Simple Analytical Expression for the Peak-Frequency Shifts of Plasmonic Resonances for Sensing"
Photonic crystal cavity (high $Q$)

$$\frac{\Delta \tilde{\omega}}{\tilde{\omega}} = \frac{-\alpha}{2V(r_0)}$$

$\alpha$ is real

$$\begin{align*}
\text{Re} \Delta \tilde{\omega} &= \text{Re} \tilde{\omega} \frac{-\alpha}{2V(r_0)} \\
\text{Im} \Delta \tilde{\omega} &= \text{Im} \tilde{\omega} \frac{-\alpha}{2V(r_0)}
\end{align*}$$

Changes in cavity loss rate follow the exact same dependence as changes in the real frequency, with the sign of the polarizability setting the sign of the change in loss rate.

$\Delta \lambda > 0$

$\Delta \omega > 0$

$\Delta \text{normalized PL Int.} > 0$

$a = 331 \text{ nm} \quad (Q = 2500)$

K. Cognée et al. (submitted)

"Mapping Complex Mode Volumes with Cavity Perturbation Theory"
Key role of the phase of the mode

\[
\frac{\Delta \tilde{\omega}}{\tilde{\omega}} \approx \frac{-\alpha \varepsilon(r_0)\tilde{E}^2}{\iiint [\varepsilon\tilde{E}^2 - \mu_0\tilde{H}^2]d^3r} = \frac{-\alpha}{2\tilde{V}(r_0)}
\]
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
Purcell factor revisited

\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re} \left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentzian} \left( \omega - \tilde{\Omega}_m \right) \left[ 1 + 2Q_m \frac{\omega - \tilde{\Omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right]
\]

"Theory of the spontaneous optical emission of nanosize photonic and plasmon resonators"
\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re} \left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentz}(\omega - \tilde{\omega}_m) \left[ 1 + 2Q_m \frac{\omega - \tilde{\omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right]
\]

\[V_1 = (-3 - 7i) \frac{\lambda^3}{10,000}\]

\[V_2 = (4 + 3i) \frac{\lambda^3}{10,000}\]

"Theory of the spontaneous optical emission of nanosize photonic and plasmon resonators"
Purcell factor revisited

\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re} \left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentzian}(\omega - \tilde{\Omega}_m) \left[ 1 + 2Q_m \frac{\omega - \tilde{\Omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right]
\]

limit of Hermitian physics \((Q \to \infty)\)

\[
\tilde{V} = \frac{\iiint (\varepsilon \tilde{E}^2 - \mu_0 \tilde{H}^2) d^3r}{2\varepsilon(r_0)\tilde{E}^2(r_0)} \to V(r_0) = \frac{\iiint [\varepsilon |\tilde{E}|^2 + \mu_0 |\tilde{H}|^2] d^3r}{2\varepsilon(r_0)|\tilde{E}(r_0)|^2}
\]

\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} \frac{Q_m}{V_m} \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentzian}(\omega - \tilde{\Omega}_m)
\]

P. Lalanne et al., Laser Photonics Rev. 12, 1700113 (2018)
"Light interaction with photonic and plasmonic resonances"
Purcell factor

\[
\frac{\gamma}{\gamma_0} = \sum_m \frac{3}{4\pi^2} Q_m \text{Re} \left( \frac{1}{\tilde{V}_m} \right) \left( \frac{\tilde{\lambda}_m}{n} \right)^3 \text{Lorentz} (\omega - \tilde{\Omega}_m) \left[ 1 + 2Q_m \frac{\omega - \tilde{\Omega}_m}{\omega} \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right]
\]

\[V_1 = (-3 - 7i) \frac{\lambda^3}{10,000}\]
\[V_2 = (4 + 3i) \frac{\lambda^3}{10,000}\]

- The contribution of each individual mode is no longer Lorentzian.
- Each individual mode does not necessarily contribute positively to the total LDOS.

"Theory of the spontaneous optical emission of nanosize photonic and plasmon resonators"
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
Strong coupling

Quantum perspective

Classical perspective
\[ \alpha(\omega) = \frac{3\pi c^3}{\omega^3} \frac{\gamma_e}{\omega_e - \omega - i\gamma_e/2} \]

\[ \tilde{E}_m, \tilde{\omega}_m \]

- **Master equation**

\[
\text{Det} \begin{pmatrix}
\tilde{\omega}_e - \omega & -\kappa_{12}(\omega) \\
-\kappa_{21}(\omega) & \tilde{\omega}_m - \omega
\end{pmatrix} = 0
\]

with \( \kappa_{12}(\omega) = \varepsilon_0 \frac{3\pi c^3}{n\omega^3} \gamma_e \) and \( \kappa_{21}(\omega) = \omega (\tilde{E}_m(r_e) \cdot u)^2 \)

- **Solution**

\[ \tilde{\omega}_\pm = \omega_e - i\frac{\gamma_e + \Gamma_m}{4} \pm \sqrt{\Omega^2 - \left(\frac{\gamma_e - \Gamma_m}{4}\right)^2} - i\Omega^2 \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \]
Strong coupling

\[ \tilde{\omega}_\pm = \omega_e - i \frac{\gamma_e + \Gamma_m}{4} \pm \sqrt{\Omega^2 - \left( \frac{\gamma_e - \Gamma_m}{4} \right)^2} - i \Omega^2 \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \]

\[ \omega_e, \gamma_e \quad \text{localized plasmon} \]

\[ \Omega_m = \omega_e, \Gamma_m \]

\[ \Omega^2 \rightarrow \Omega^2 \left[ 1 - i \frac{\text{Im}(\tilde{V}_m)}{\text{Re}(\tilde{V}_m)} \right] \]

Coupling strength (or Rabi frequency)

P. Lalanne et al., Laser Photonics Rev. 12, 1700113 (2018)
“Light interaction with photonic and plasmonic resonances”
Introduction

1. The non-Hermitian nature of quasinormal modes

2. Numerical tests & examples

3. Physics of resonance
   - Cavity Q’s
   - Cavity V’s
   - Cavity perturbation theory
   - Purcell effect
   - Strong coupling
   - Measurement of complex V’s
Measurements of complex mode

Fabrication: Andrea Fiore (Eindhoven Univ. Technology)

Near-field exp.: Massimo Gurioli (LENS-Florence)
Measurements of complex mode volumes

"Mapping Complex Mode Volumes with Cavity Perturbation Theory"
\[ \frac{\Delta \tilde{\omega}}{\tilde{\omega}} = \frac{-\alpha \varepsilon(\mathbf{r}_0)\tilde{E}^2}{\iint[\varepsilon \tilde{E}^2 - \mu_0 \tilde{H}^2]d^3\mathbf{r}} \equiv \frac{-\alpha}{2\tilde{V}(\mathbf{r}_0)} \]

(a) Re\(\left(\frac{a^3}{2\tilde{V}}\right)\)

(b) Im\(\left(\frac{a^3}{2\tilde{V}}\right)\)

K. Cognée et al., submitted.
"Mapping Complex Mode Volumes with Cavity Perturbation Theory"
Conclusion

Efficient QNM solvers

Simple closed-form expression for the QNM-expansion

Rapid recent progress towards a new path in nanoresonator modeling:
- fast, accurate
- analytical with respect to the frequency (unique)
- with transparent physics (unique)
- Review article is available (Laser Photonics Rev. 2018)
Conclusion

Efficient QNM solvers

Simple closed-form expression for the QNM-expansion

Original New applications:
- Analysis of resonance with spatial dispersion
- Coupled-QNM theory of nanoresonators (molecular plasmonic, plasmonic-photonic hybrids ...)
- Resonance with back-action (NL optics, trapping ...)
- Cavity quantum electrodynamics